Lecture 11

- Finish TU
- nonbip. matching polytope
- Next time: flows.

But can check in polynomial time!

Total unimodularity

Recall: A T.U. $\Leftrightarrow$ all subdeterminants in $\{0,-1,+1\}$.

↳ checking T.U. naively needs $2^n \cdot 2^m$ determinants.

The point: when does an I.P. have same solutions as its L.P. relaxation?

e.g

$$z_{IP} := \min \; c^T x \quad Ax = b, \quad x \geq 0, \quad \boxed{x \in \mathbb{Z}^n} \quad \text{(I.P.)}$$

vs

$$z_{LP} := \min \; c^T x \quad \underbrace{\boxed{Ax = b, \quad x \geq 0}}_{P} \quad \text{(LP.)}$$

<u>Always</u>: $z_{IP} \geq z_{LP}$.

<u>Main result from last lecture:</u>

If A is TU then $z_{IP} = z_{LP}$; $b \in \mathbb{Z}^m$

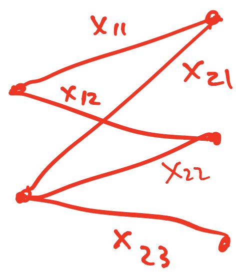
In particular, $P = \{x : Ax = b, x \geq 0\}$ <u>integral</u>.

<u>Example</u> <u>bipartite matching</u>.

polytope of "fractional matchings" we used for min-weight-perfect-matching:

Let (U, V) be bipartition.

$$P = \Big\{ x \in \mathbb{R}^E : \sum_{(i,j) \in E} x_{ij} = 1 \quad \forall i \in U$$



$$\sum_{(i,j) \in E} x_{ij} = 1 \quad \forall j \in V$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in E \Big\}$$

$$:= \{ x \in \mathbb{R}^E : Ax = b, \ x \geq 0 \}.$$

Integral points in P = perfect matchings in G!

Recall: Lecture on bipartite matching

$\Rightarrow$ P is <u>integral</u>.* i.e.

<u>THM</u>: (MWPM THEOREM)

$$\boxed{\text{MWPM} = \min_{\substack{M \\ \text{perfect} \\ \text{matching} \\ \text{in } G}} \sum_{(i,j) \in M} c_{ij} = \min\{c^T x : x \in P\} \qquad \text{IP} = \text{LP}.}$$

* technically only showed for G = complete bipartite, but also true for <u>any</u> bipartite G.

<u>Another</u> way to show it:

<u>**Theorem**</u>: The matrix A is totally unimodular.

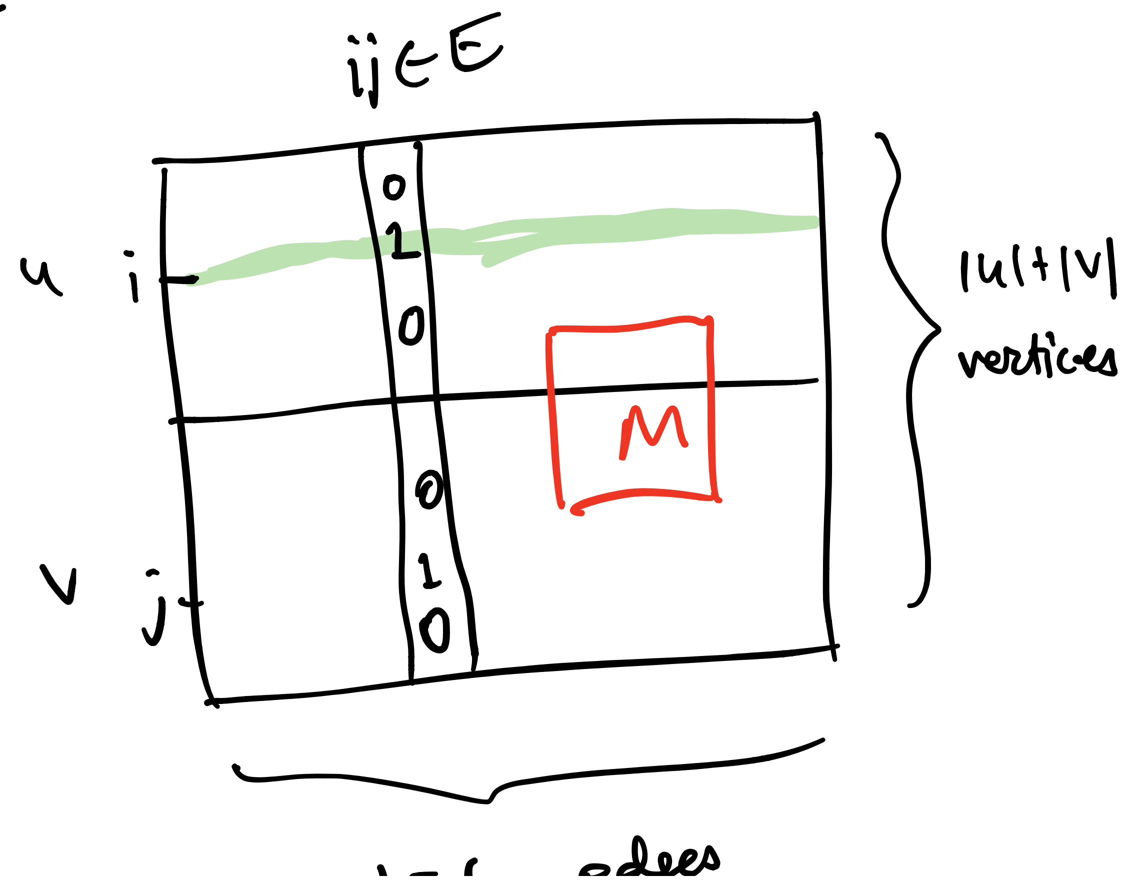
Cor: MWPM THEOREM.

Proof: What's A look like?

A^T is <u>incidence matrix</u> of G.

<u>i.e.</u>

$A =$



$|E|$

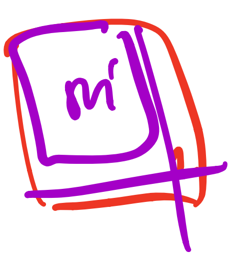
- To show A is TU, consider square submatrix M & look at cases:

  1) if M has 0 row/col,

     $\boxed{\det M = 0}$

  2) if M has row/col w/ only one 1,

     expand down that row/col reduce to smaller submatrix M'



3) M has $\geqslant 2$ nonzero entries per row & col.

$\Rightarrow$ M has exactly 2 nonzero entries per column

$$M \qquad \left[\begin{array}{ccccc} 1 & 1 & \cdot & 1 & \\ & & & & 1 \\ \hline & & 1 & 1 & \\ 1 & & & & 1 \end{array}\right] \begin{array}{l} \Big\} := u_0 \\ \\ \Big\} := v_0 \end{array}$$

$$\mathbb{1}_{u_0} = \begin{bmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \begin{array}{l} u_0 \\ \\ v_0 \end{array} \qquad \mathbb{1}_{v_0} = \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} \begin{array}{l} u_0 \\ \\ v_0 \end{array}$$

$$\boxed{\mathbb{1}_{U_0}^T M = \mathbb{1}}$$

(add up rows of M in U_0, get $\mathbb{1}^T$). Similarly

$$\boxed{\mathbb{1}_{V_0}^T M = \mathbb{1}}$$

$$\Rightarrow \left(\mathbb{1}_{U_0}^T - \mathbb{1}_{V_0}^T\right) M = 0$$

rows <u>not</u> lin indep.

$$\Rightarrow \boxed{\det(M) = 0.} \qquad \square$$

Neat side note:

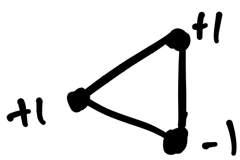
$m \times n$

Def: discrepancy of $A \in \mathbb{R}^{m \times n}$ is

$$\min_{x \in \{\pm 1\}^n} \|Ax\|_\infty = \min_{x \in \{\pm 1\}^n} \max_{i \in m} |(Ax)_i|.$$

How well A can be "balanced".

E.g. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has discrepancy 2 because of $x = (-1, +1, +1)$



and some two entries have same sign.

Fact: A is T.U. $\Leftrightarrow$ all submatrices of A have discrepancy ≤ 1

T.U. matrices are highly "balanceable".

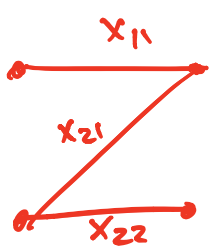
(Non-bipartite) Matching polytope

We saw (lecture 1, prev. example) that if $G = ((U,V), E)$ bipartite, then the convex hull of p.m.'s is

$$P = \Big\{ x \in \mathbb{R}^E : \sum_{(i,j)\in E} x_{ij} = 1 \quad \forall i \in U$$

"degree constraints".

$$\sum_{(i,j)\in E} x_{ij} = 1 \quad \forall j \in V$$

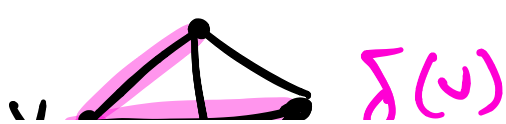


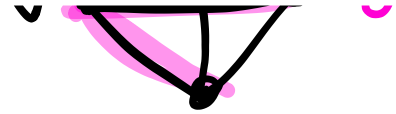
$$x_{ij} \geq 0 \quad \forall (i,j) \in E \Big\}$$

But for <u>nonbipartite</u>?

- Degree constraints enough?

Def $\delta(v) = \{e : v \in e\}$

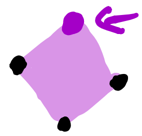




not perfect

Could we have conv(matchings) =

$$P = \left\{ x \in \mathbb{R}^E : \sum_{e \in \delta(v)} x_e \le 1 \ \forall v \in V, \quad x_e \ge 0 \ \forall e \in E \right\} ?$$



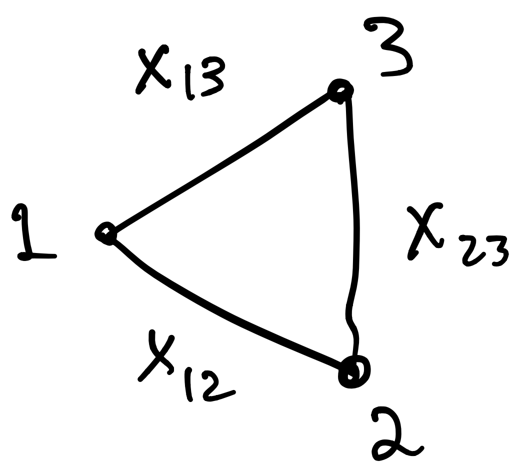
← $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

← matchings.

Constraining x to be integral does yield matchings!

No: E.g.

IP ≠ LP for P as above!



- $x_{12} = x_{13} = x_{23} = \frac{1}{2}$ feasible; $\in P$
- Sum is $\frac{3}{2}$ but every matching has sum ≤ 1.

$x \notin \text{conv(matchings)}$

Need another constraint:

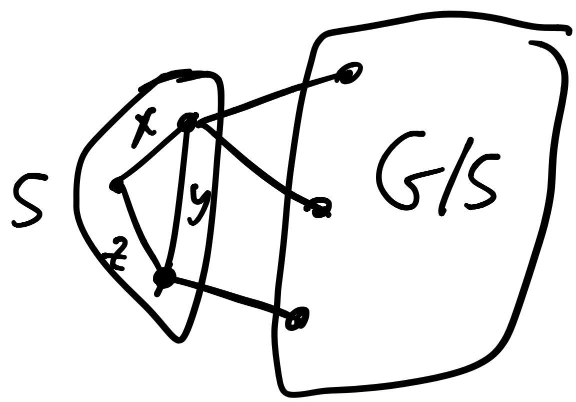
"ODD SET CONSTRAINT"

max # edges of M in odd set S

If $|S|$ odd, then

$$\sum_{e \in E(S)} x_e \le \boxed{\frac{|S|-1}{2}}$$

e.g.



$x+y+z \le 1$

$\frac{3-1}{2}$.

ODD set constraints hold for $x \in \mathrm{conv}(\text{matchings!})$.

THM (Edmonds) Let

$\boxed{A \text{ not TU}}$

$X = \{1_M : M \text{ matching in } G\}$.

← any matching.

Then $\mathrm{conv}(X) = P$ where

$$P = \left\{ x : \begin{array}{ll} \sum_{e \in \delta(v)} x_e \le 1 & \forall v \in V. \\ \sum_{e \in E(S)} x_e \le \frac{|S|-1}{2} & \forall S \subseteq V \\ & |S| \text{ odd} \\ x_e \ge 0 & \forall e \in E. \end{array} \right\}$$

degree constraints

odd set constraints.

nonnegativity constraints

$P \cap \left\{ \sum_{e \in E} x_e = \frac{|V|}{2} \right\}$ = "or set degree constraints to equality"

= conv(perfect matchings).

Proof: Idea: show they have the same facets.

$\boxed{\le}$ conv(X) $\subseteq$ P ("showed" before)

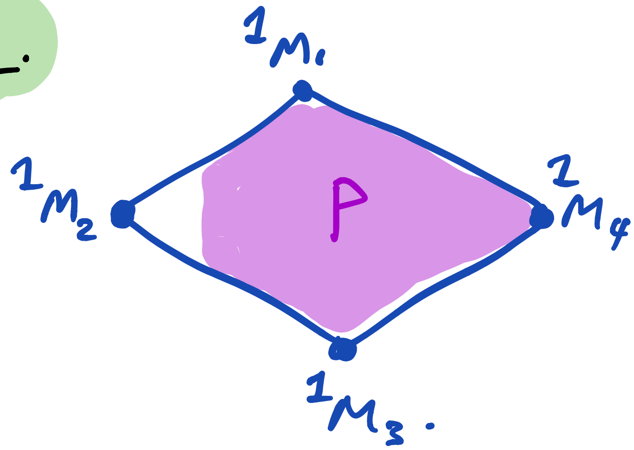
$\boxed{\ge}$ To show conv(X) $\supseteq$ P, show every facet of conv(X) comes from inequality of P.

($\Rightarrow$ P has more constraints than conv(X)

$\Rightarrow$ containment $\supseteq$).

*

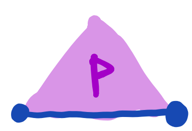
E.g.



$\mathrm{conv}(X) \supseteq P$.

* Caveat: need $\mathrm{conv}(X)$ full-dimensional for this proof strategy to work.

e.g.



Every facet of $\mathrm{conv}(X)$ is ineq of P, but $\mathrm{conv}(X) \not\supseteq P$.

Showing 2:

- **Step 1**: Show $\boxed{\dim \mathrm{conv}(X) = |E|}$.

$\subseteq \mathbb{R}^E$

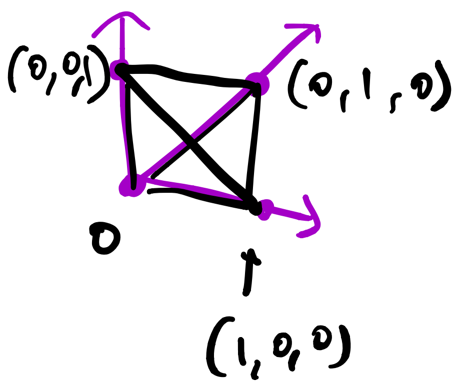
Recall:

$$\dim \mathrm{conv}(X) = \left(\begin{array}{c}\text{max \# of affinely} \\ \text{indep. points in } \mathrm{conv}(X)\end{array}\right) - 1$$

need $|E|+1$ affinely independent points!

Affine independence refresher

$\mathbb{R}^E$ $|E| = 3$.



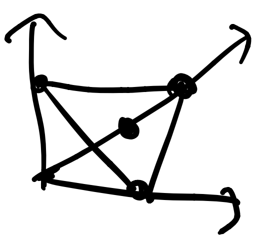
X affinely independent

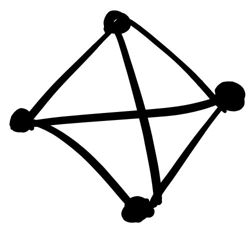
$\iff$

smallest affine space containing X has dimension $= |X| - 1$.

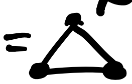
$\iff$

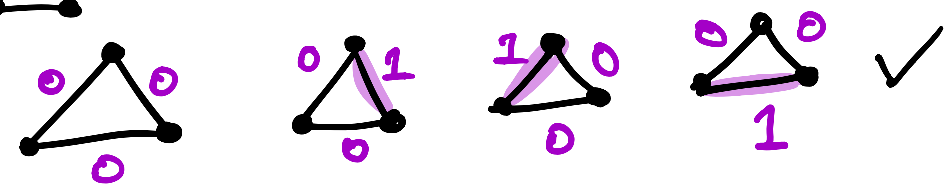
$\text{conv}(X)$ is a simplex.





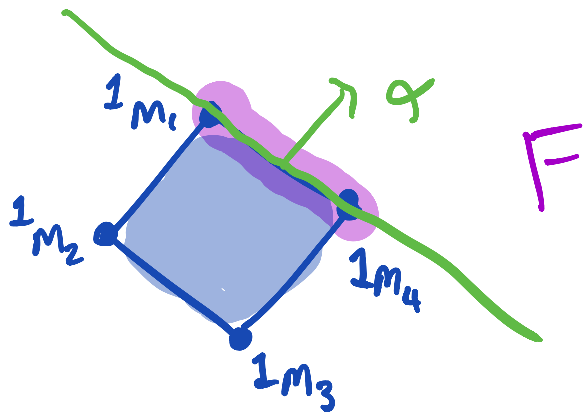
in $\mathbb{R}^3$ tetrahedron.

e.g. $G =$  $1_\emptyset$, $\{1_{\{e\}} : e \in E\}$ $0, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$



- **Step 2:** Now consider face F of $\text{conv}(X)$ from inequality $\alpha^T x \le \beta$.

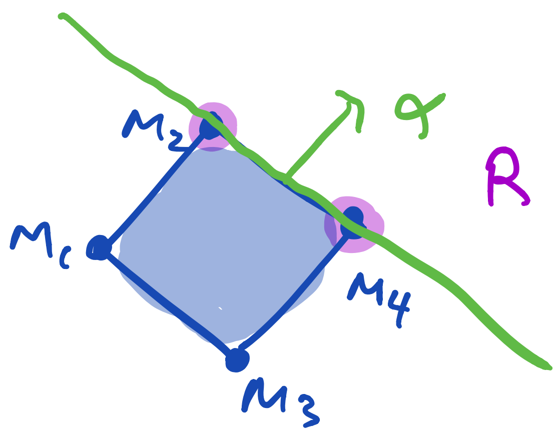
  valid, i.e. holds for all 1_M.



- Need to show F contained in face from inequality of P.
  (i.e. either degree constraint, odd set constraint, or nonnegativity constraint.)

Note: $F = \mathrm{conv}(R)$ where

$R = \{x \in X : \alpha^T x = \beta\} :=$ "extremal matchings"



- Calling elts. of R "matchings", conflating $\mathbb{1}_M \leftrightarrow M$ are abuses of notation, but we do it anyway.
- Enough to show all elements of R are tight for some ineq. of P.
- If R empty, done. Assume not.

- **Case (a):** α has negative entry α_e.

$\Rightarrow x_e = 0 \quad \forall x \in R$

(else setting $x_e=0$ increases $\alpha^T x$, violates extremality of x).

$\Rightarrow F \subseteq$ face from nonnegativity constr. $x_e \geq 0$.

assume $\alpha \geq 0$ for remaining cases.

- **Case (b):** Some vertex v covered by every $x \in R$, i.e.

$$\boxed{\sum_{e \in \delta(v)} x_e = 1 \quad \forall x \in R.}$$

$\Rightarrow F \subseteq$ face from degree constraint

$$\boxed{\sum x_e \leq 1}$$

$e \in \delta(v)$

For final case:

Assume $\forall v$, is extremal matching $M_v \in R$ not covering v.

- Case (c):
  - Let E_+ be edges where $\alpha > 0$, i.e.

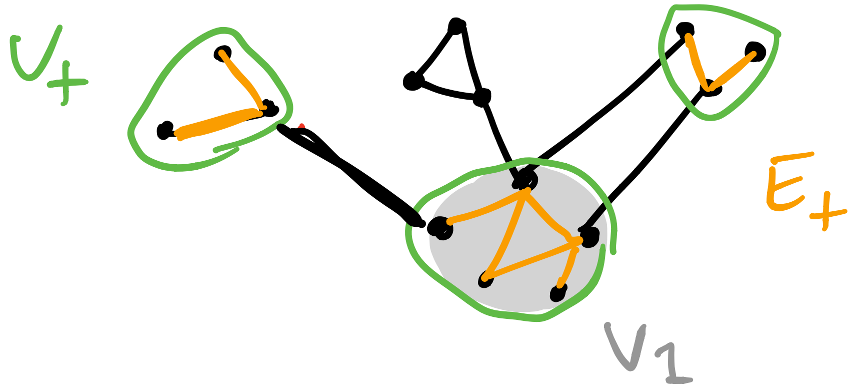
$$E_+ = \{e \in E : \alpha_e > 0\};$$

Case (a) $\Rightarrow$ $\alpha_e = 0$ for $e \in E \setminus E_+$.

  - Let V_+ = vertex set of E_+,

(V_1, E_1) = any connected component of (V_+, E_+).

e.g.

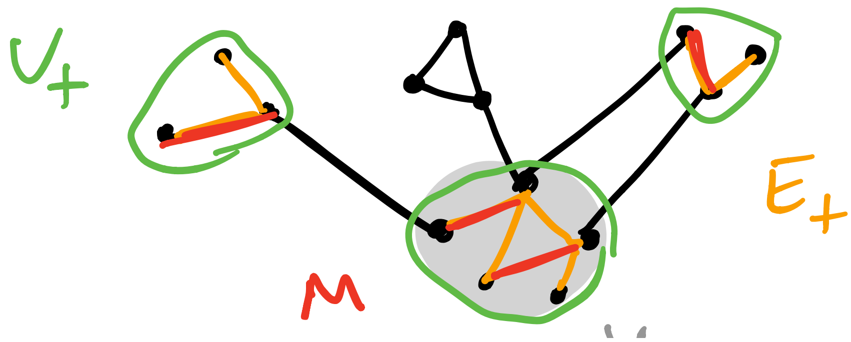


Claim: F contained in face from odd set constraint w/ $S = V_1$.

i.e. $\boxed{\sum_{e \in E(V_1)} x_e = \frac{|S|-1}{2}}$ $\forall x \in R$

equiv: all $M \in R$ have $\boxed{\frac{|S|-1}{2}}$ edges in V_1.

e.g.



v1

Idea of proof of claim:

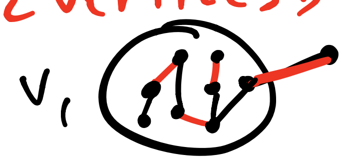
Show

* no extremal matching $M \in R$ missing some two verts. $u, v \in V_1$

why is * enough?

* $\Rightarrow$ (i): any matching $M \in R$ missing ≥ 1 vertex of V_1 can't have edges e departing V_1

M' still extremal b/c $\alpha_e = 0$. $\rightarrow$ (removing $e \rightsquigarrow M'$ missing 2 vertices, contradicts *).

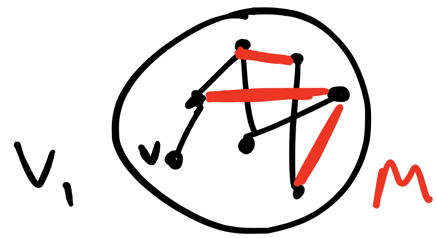


doesn't happen

(i.e. M near perfect in V_1)

(i) $\Rightarrow$ (ii): $|V_1|$ odd, because $\exists\, M \in R$ missing some $v \in V_1$; (from case (b))

& M near perfect in V_1



(i),(ii) $\Rightarrow$ $\boxed{\text{Every extremal } M \text{ has } \frac{|V_1|-1}{2} \text{ edges of } E_1;}$

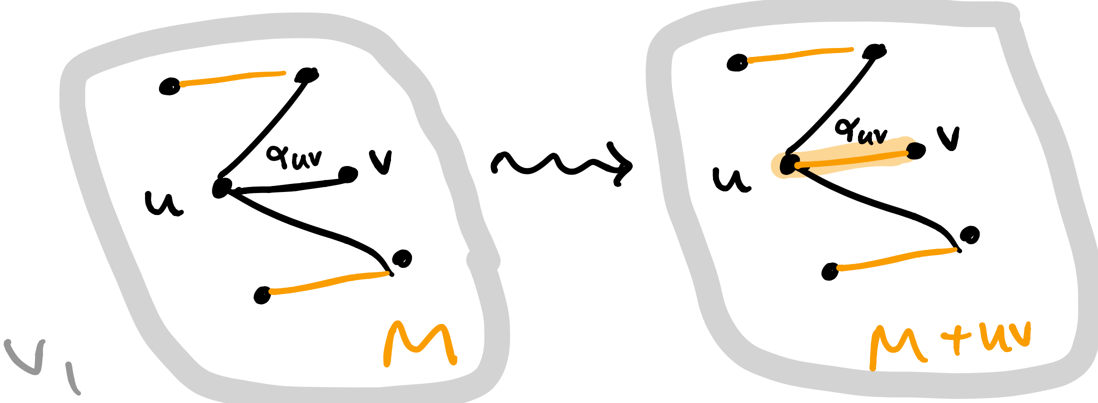
($|V_1|$ odd $\Rightarrow$ every M missing some $v \in V_1$ $\underset{(i)}{\Longrightarrow}$ M near perfect in V_1.) done! (modulo *).

Proof of * :

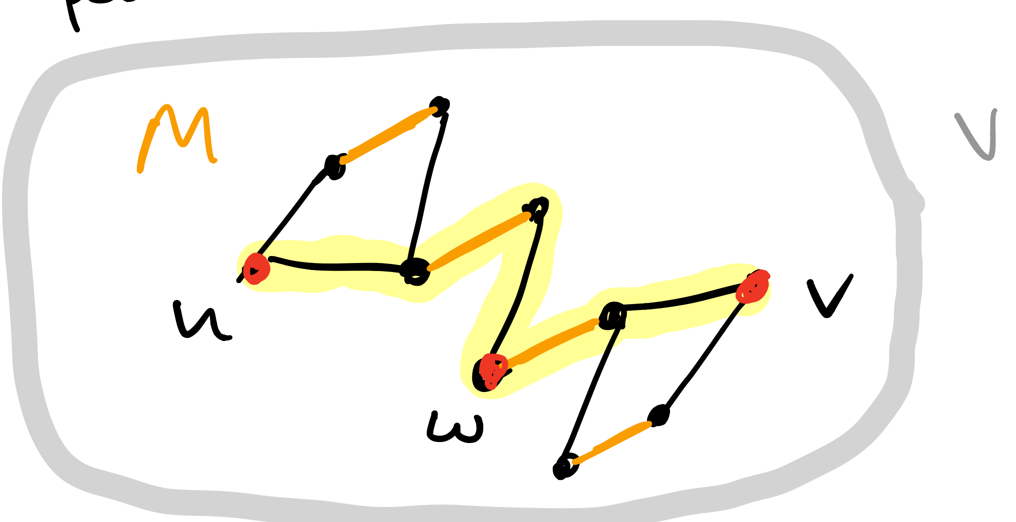
- Suppose not: Among elts of R (extremal matchings) excluding some two verts $u, v \in V_1$ let $M \in R$ be matching missing $\boxed{u, v \text{ that are closest in } (V_1, E_1)}$ (also may be missing other vertices)

(will build new matching M_1 missing even closer vertices $\Rightarrow$ contradiction.)

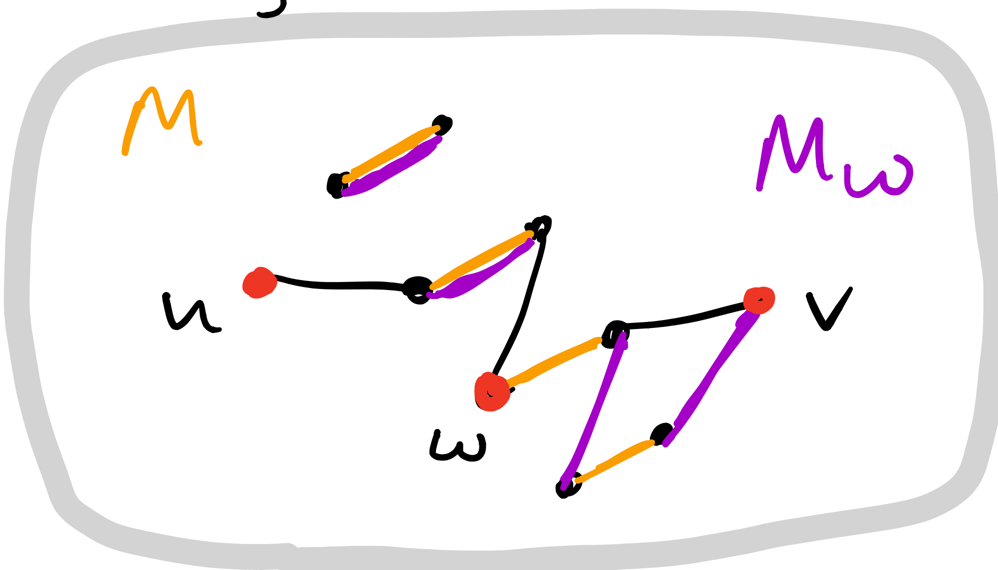
- If dist $= 1$, then $(u,v) \in E$, $\subseteq E_+$
  $\Rightarrow$ $M + uv$ violates $\alpha^T x \le \beta$ (increases $\alpha^T x$ b/c $\alpha_{uv} > 0$.)



- Thus, distance ≥ 2. Let $w \notin \{u,v\}$ on shortest u-v path.

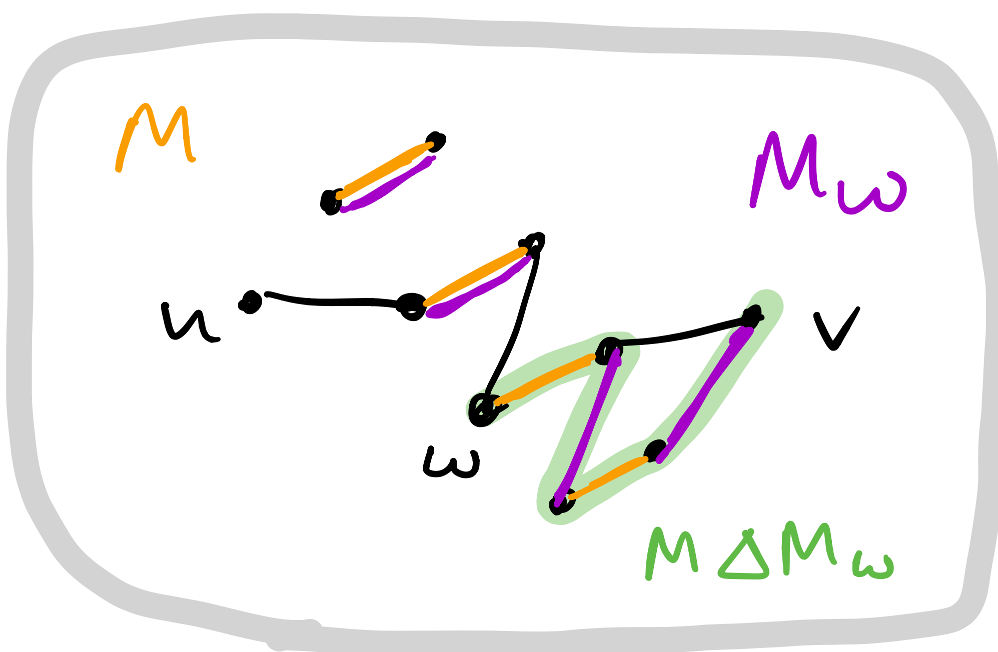


- Case(b) $\Rightarrow \exists\ M_\omega \in K$ missing ω.

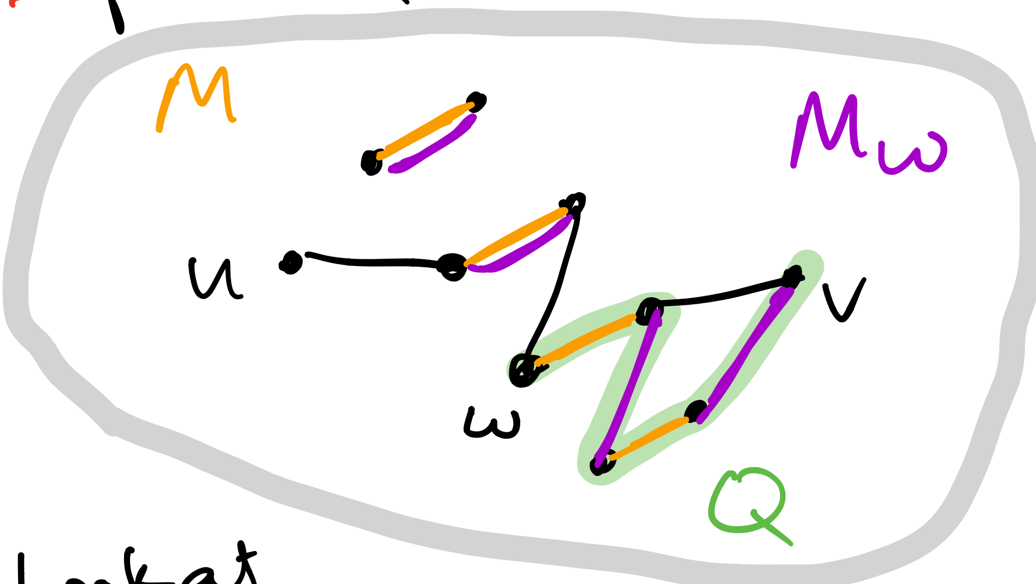


- look at $M_\omega \triangle M$.

  is symmetric diff of matchings.

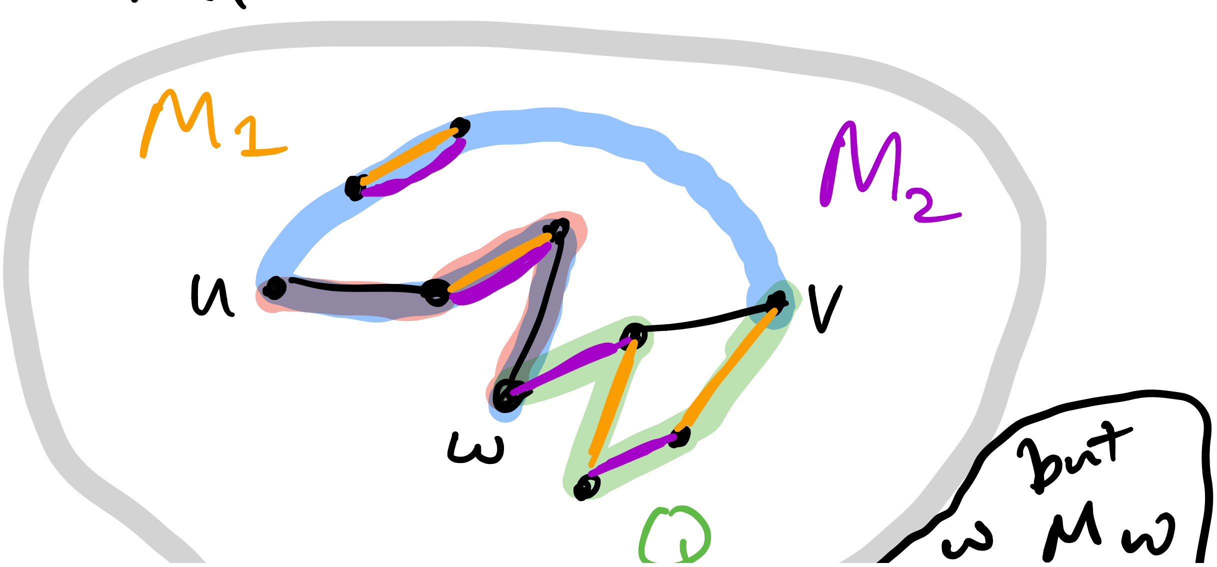


- $M_w \Delta M$ is union of alternating! paths & cycles, contains some alt. path Q ending at w.



- Look at

$M_1 = M \Delta Q$, $M_2 = M \Delta Q$.



$\boxed{Q \neq \emptyset \text{ because } M \text{ covers } \quad w,\ \text{not.}}$

- M_1, M_2 matchings $\Rightarrow \sum_{e \in M_i} \alpha_e \leq \beta.$

for $i \in \{1,2\}$, but

$$\boxed{\underbrace{\sum_{e \in M_1} \alpha_e}_{\leq \beta} + \underbrace{\sum_{e \in M_2} \alpha_e}_{\leq \beta} = \sum_{e \in M} \alpha_e + \sum_{e \in M_w} \alpha_e = 2\beta}$$

$\leftarrow$ BOTH EQUAL!!

(b/c M_1, M_2 counts / double counts edges the same way as M, M_w.

b/c just swapped some differing edges.)

$$\Rightarrow \boxed{\sum_{e \in M_i} \alpha_e = \beta.}, \ i \in \{1,2\}.$$

$\Rightarrow$ Both $M_1, M_2 \in R$ (extremal!)

- But M_1 doesn't cover w;

& doesnt cover one of u or v
(at least one of u, v wasnt part of Q.)

$\Rightarrow$ M_1 is missing two vertices closer than u, v; $\square$

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why doesnt M_1 cover both u, v?

Q cant contain both u, v

- b/c ~~it~~ can only contain them as endpoints
- ~~b/c~~ Q alternates in M
-

which doesn't cover u, v.

- one endpt is w.